Modified Borel summation of divergent series and critical-exponent estimates for an *N*-vector cubic model in three dimensions from five-loop ϵ expansions

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An approach to summation of divergent field-theoretical series is suggested. It is based on the Borel transformation combined with a conformal mapping and does not imply the exact asymptotic parameters to be known. The method is tested on functions expanded in their asymptotic power series. It is applied to estimating the critical exponent values for an *N*-vector field model, describing magnetic and structural phase transitions in cubic and tetragonal crystals, from five-loop ϵ expansions. [S1063-651X(98)04310-4]

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Recently Kleinert and Schulte-Frohlinde calculated renormalization group (RG) functions for the cubic model in $4-\epsilon$ dimensions within the five-loop approximation of the renormalized perturbation theory [1]. The critical (marginal) dimensionality N_c of the order parameter field found on the basis of those expansions turned out to be smaller than 3 in three dimensions. This means that the critical behavior of the model should be controlled by the cubic fixed point with a specific set of critical exponents. Thus, calculation of the critical exponent values for the cubic model is an actual problem. Since the model of concern describes the critical behavior of cubic and tetragonal crystals undergoing magnetic and structural phase transitions, critical exponent estimates can be compared with experimental data.

The field-theoretical RG series are known to be badly divergent. To extract the physical information relevant to predicting the critical behavior of real systems, they should be processed by a proper resummation procedure. The series for N_c obtained in Ref. [1] proved to be alternating in signs that allowed to resum it by the simple Padé method. At the same time, the critical exponent series are irregular and cannot be processed by the ordinary (Padé, Padé-Borel, Padé-Borel-Leroy) techniques. A more sophisticated method of the Borel transformation combined with a conformal mapping, although regarded as the most universal procedure, is inapplicable as well, because it requires knowing the exact values of the asymptotic parameters characterizing the high-order behavior of the series. Nowadays those parameters have been evaluated for the simplest case of the $O(N)$-symmetric models only [2,3], and calculating them for anisotropic models is a most difficult problem as yet unsolved. As an exception one can mention the anisotropic quartic quantum oscillator, which represents a one-dimensional φ^4-field theory with the cubic anisotropy. For the perturbation expansion of the ground-state energy of this system, the asymptotic parameters were found in Ref. [4].

The aims of the present work are as follows. (i) To suggest an approach to summation of the divergent field-theoretical series, which is based on the Borel transformation combined with a conformal mapping, but which does not involve the exact values of the asymptotic parameters. This approach, being the result of a computer study of the Borel summation procedure, is tested on functions expanded in their asymptotic power series, on calculation of the ground state of the unharmonic oscillator, and on estimation of the critical exponent values for the basic models of phase transitions. (ii) To obtain critical exponent estimates for the cubic (simplest anisotropic) model in three dimensions from the record five-loop ϵ expansions [1], using the developed technique.

We would like to emphasize that the problem of processing divergent series arises in various fields of physics where the perturbation theory is employed but the parameter of expansion is not small. So, developing a resummation procedure that could be effective where conventional methods fail is of general interest.

A modification of the Borel procedure via a conformal mapping was introduced in Ref. [5] and used for processing series $\Sigma_k f_k g^k$, whose coefficients at large order k behaved as $k!k^{b_0}(-a_0)^k$. To the series $\Sigma_k f_k g^k$ the function

$$F(g;a,b)=\int_0^\infty e^{-x/ag}\left(\frac{x}{ag}\right)^b d\left(\frac{x}{ag}\right)B(x) \qquad (1)$$

is associated. The Borel transform $B(x)$ is the analytical continuation of its Taylor series $\Sigma_k[f_k/a^k\Gamma(b+k+1)]x^k$ absolutely convergent in the unity circle. Usually $a=a_0$, whereas the parameter b may be related to the exact asymptotic value b_0 in a variety of ways [5,6]. Conformal mapping $\omega=\sqrt{x+1}-1/\sqrt{x+1}+1$ transforms the cut-plain $C\setminus[-1,-\infty)$ onto the unity circle, and the semiaxis $[0,\infty)$, the domain of integration, goes over into the interval $[0,1)$. Supposing $B(x)$ may be continued over the cut-plain, the composite function $B(x(\omega))$ is holomorphic within the unity

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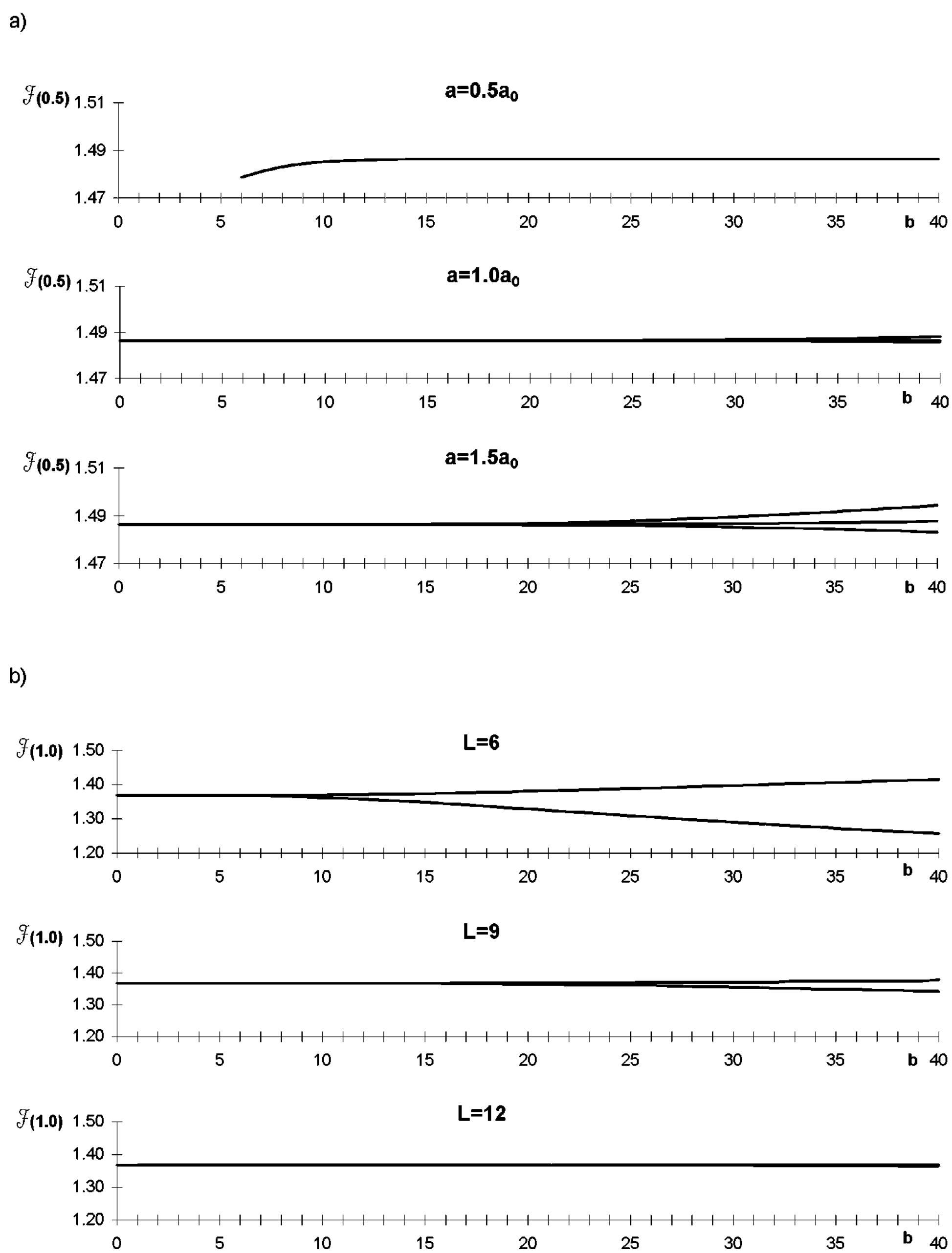

FIG. 1. (a) Graphs of dependence of $\mathcal{F}_L^i(0.5;a,b)$, $L=13$, on parameter b for some values of parameter a. (b) Graphs of dependence of $\mathcal{F}_L^i(g;1,b)$ on parameter b in sixth, ninth, and twelfth order in g for $g=1$.

circle and its Taylor series admits integration before summation. In order to eliminate the possible singularity of $B(x(\omega))$ at $\omega=1$, an additional parameter λ was introduced in Ref. [6] and used in Refs. [7,8]: $B(x(\omega))=A(\lambda,\omega)/(1-\omega)^{2\lambda}$. It is chosen from the condition of the most rapid convergence of the series

$$F(g;a,b)=\sum_{k=0}^{\infty} A_k(\lambda)\int_0^{\infty} e^{-x/ag}\left(\frac{x}{ag}\right)^b \times d\left(\frac{x}{ag}\right)\frac{\omega^k(x)}{[1-\omega^k(x)]^{2\lambda}}, \qquad (2)$$

that is, from minimizing the quantity $|1-F_L(g;a,b)/F_{L-1}(g;a,b)|$, where L is the step of truncation and $F_L(g;a,b)$ is the L-partial sum of the series (2). In practice, one deals with a piece of the series only, where the asymptotic regime might not be reached. For this reason, in Refs. [5,9,10] the parameter b was varied in a neighborhood of b_0. We believe that in the case of the unknown exact asymptotic value a_0 similar manipulations may apply to the parameter a as well.

Our approach to using the procedure described above consists in the following. While processing the asymptotic series of *a priori* given functions, we have revealed that the quantity $F_L(g;a,b)$ remains stable as the parameters a and b vary in a wide range. With the ‘‘number of loops’’ increasing, the (a,b) dependence becomes weaker and weaker, and the difference between $F_L(g;a,b)$ and the exact value $F(g)$ can be made small with any accuracy. This observation enables us to employ the Borel transformation with a conformal map-

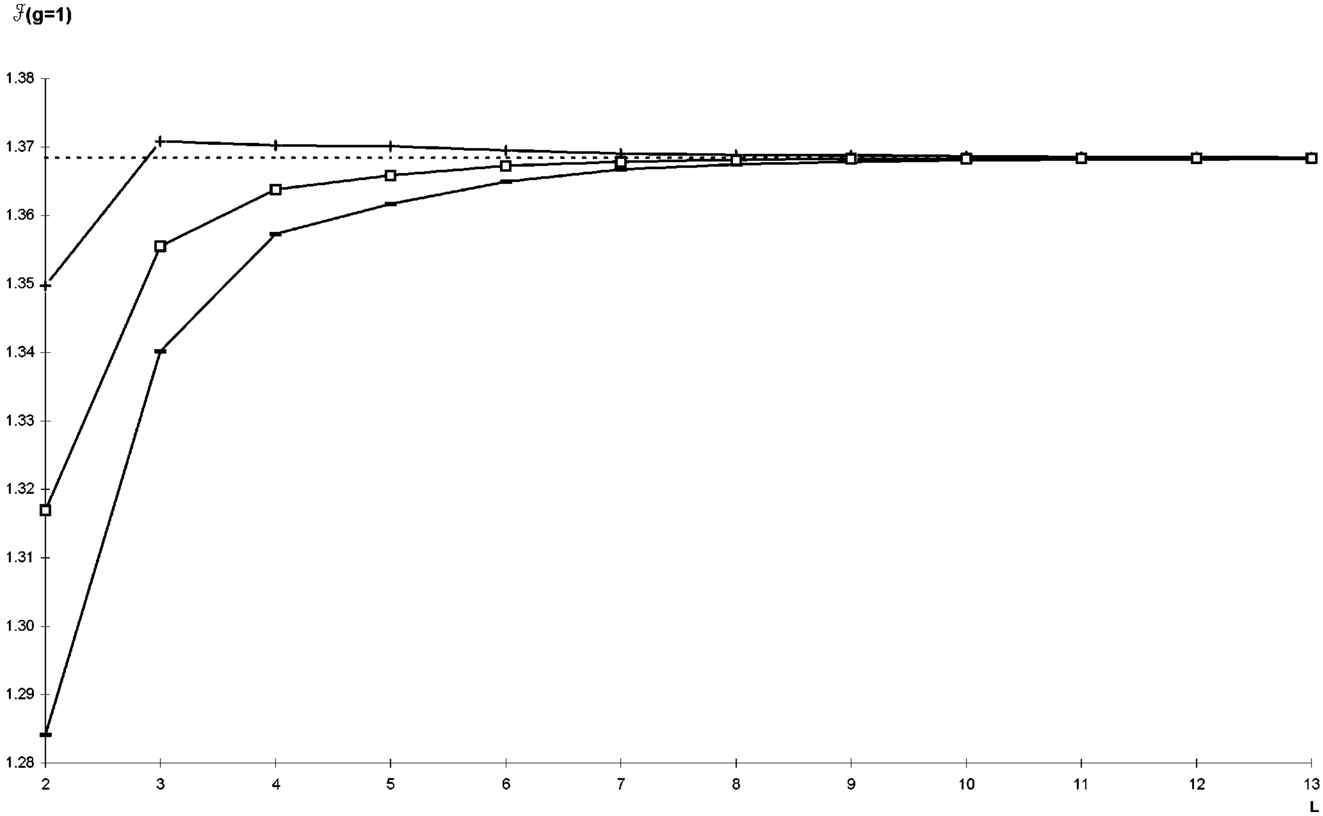

FIG. 2. Convergence of the numerical estimates of the function $\mathcal{F}(g)=\int_{-\infty}^{+\infty}e^{-x^2-gx^4}dx$ depending on the approximation order L for $g=1$. The dashed line corresponds to the exact value 1.368 427 . . . . The upper and lower broken lines yield, respectively, the upper and lower boundaries for the estimate. The best estimate is given by the middle line.

ping for processing series whose exact asymptotic behavior is unknown. Thus, we postulate the stability of the result of the processing with respect to variation of a and b as a basic principle underlying our approach.

Let us demonstrate how this principle works on some simple examples. At first, consider a model function

$$\mathcal{F}(g)=\int_{-\infty}^{+\infty}e^{-x^2-gx^4}dx\sim\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(2k+\frac{1}{2}\right)}{k!}g^k$$

whose coefficients f_k behave as $[(-4)^k/\sqrt{2\pi}](k!/k)$ at large k. For each a and b in Eq. (2) we find the set of values $\{\lambda_{\min}^i(a,b)\}$ at which the quantity $|1-\mathcal{F}_L(g;a,b)/\mathcal{F}_{L-1}(g;a,b)|$ as a function of λ reaches its local minima. To each element of that set corresponds the particular value $\mathcal{F}_L^i(g;a,b)$. It can be shown that taking into account relative contributions of lower order terms of the series ($k<L$) when optimizing the λ parameter just weakly affects the final result, therefore it is sufficient to minimize the relative contribution of the last accounted term only. In Fig. 1(a), a few curves $\mathcal{F}_{13}^i(0.5;a_0,b)$ are presented, which, as calculations show, deviate from the exact number $\mathcal{F}(g)=1.486\,310\,82\ldots$ by no more than 5×10^{-5} within the range $0\leqslant b\leqslant18$.

For the function $\mathcal{F}(g)$ the exact asymptotic value of a is $a_0=4$. As the parameter a shifts from a_0, the picture does not change qualitatively, although at large numbers one can see a reduction of the b-stability interval, as displayed in Fig. 1(a). For small a, the set $\{\lambda_{\min}^i(a,b)\}$ is just empty within the interval [0,6]. In all cases the lines in their stability domains are at the same level with the above mentioned accuracy. So, on the graph of $\mathcal{F}_{13}^i(g;a,b)$ depending on two parameters a and b there would be a horizontal part—a plateau rapidly destroyed at its boundaries.

The size of the plateau depends essentially on g. When g gets smaller, the stability domain is enlarged. That is in accord with intuitive expectations that the decrease of the expansion variable should result in better convergence. So, the calculations show that for the family of five curves $\mathcal{F}_{13}^i(0.1;a_0,b)$, the relative error $|\mathcal{F}_{13}^i(0.1;a_0,b)-\mathcal{F}(0.1)/\mathcal{F}(0.1)|$ does not exceed 2×10^{-6} within the interval $0\leqslant b\leqslant25$. On the contrary, the growth of g leads to a reduction of the stability domain. Let us follow how its size

TABLE I. Numerical estimates for the anharmonic oscillator ground-state energy at $g=1$.

L	8	9	10	11	12	Exact value
$\mathcal{E}(1)$	1.392376	1.392357	1.392344	1.392349	1.392351	1.392352

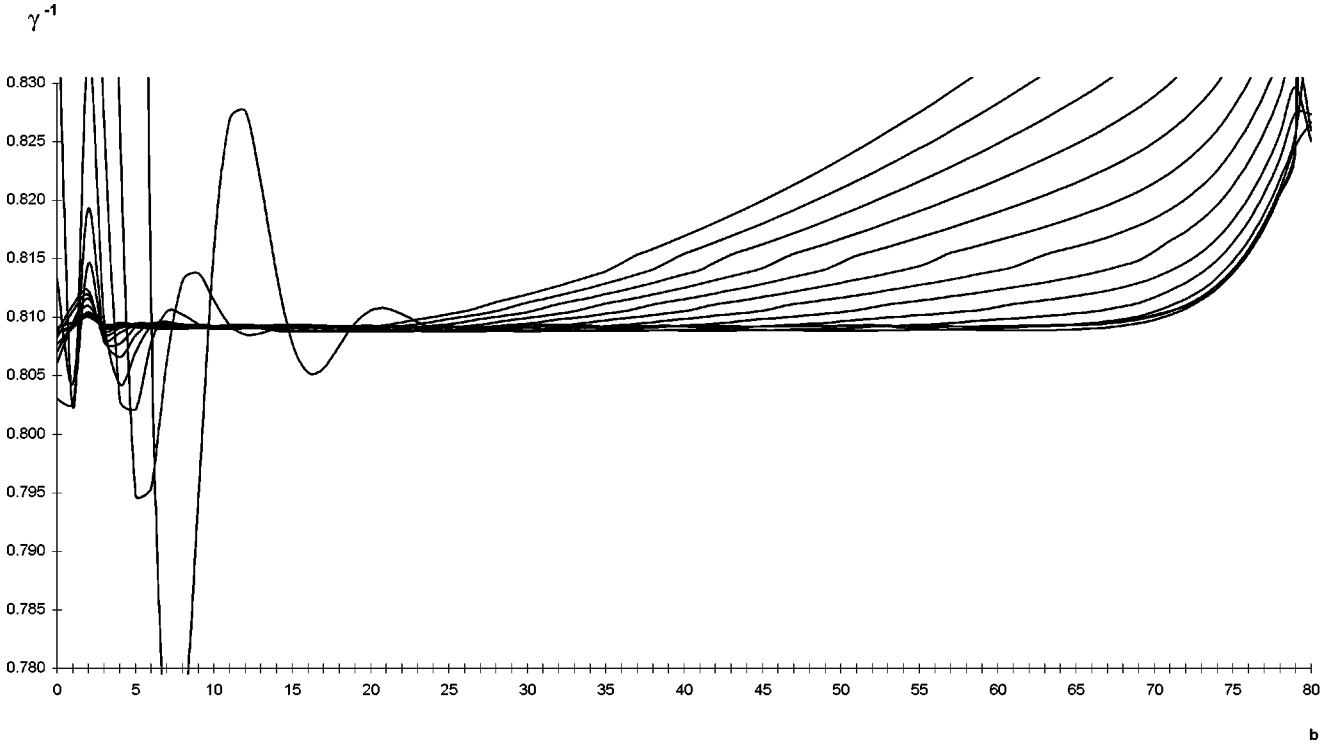

FIG. 3. Curves demonstrating dependence of the result of processing the critical exponent γ^{-1} on parameter b for various values of a for the cubic model at $N=2$ in the five-loop approximation.

changes from loop to loop, assuming $g=1$. Presented in Fig. 1(b) are the most stable curves $\mathcal{F}_L^i(1.0;a_0,b)$ for $L=6,9,12$. As seen from the graphs, by accounting for more terms of the expansion the behavior of the curves improves, and the dispersion of the values goes down. The graph illustrating convergence of numerical estimates of $\mathcal{F}(g)$ for $g=1$ depending on the truncation number L is depicted in Fig. 2.

When processing functions $\int_0^\infty e^{-x}(x\partial_x)^{b_0}(1/1+gx)\,dx \sim \Sigma_{k=0}^\infty(-1)^k k! k^{b_0} g^k$, where $b_0 \geqslant 0$, we observed a similar stability with respect to a and b. We also processed the six-loop pieces of RG series for the critical exponents of the $O(N)$-symmetric model in three dimensions and found very weak dependence of the output on the transformation parameters. The numerical values of the critical exponents computed proved to be in good agreement with those of Ref. [5].

Using the proposed summation method for processing the series of the ground-state energy $\mathcal{E}(g)$ of the anharmonic oscillator [11] with the Hamiltonian $H=x^2+gx^4$, we observed the same behavior of the corresponding curves as for the model functions considered above. In Table I we present the estimates of $\mathcal{E}(g)$ at $g=1$ depending on the length of the series. Note that for $L=8$ our estimate is closer by one order to the exact value than the number $1.391\,655 \pm 0.004\,562$ found in Ref. [12] on the basis of Wynn's ϵ algorithm.

The fulfilled numerical analysis allows us to apply the introduced summation method to find numerical estimates of the critical exponents for the $(4-\epsilon)$-dimensional cubic model [1] in three dimensions $(\epsilon=1)$. The b dependence of the results of processing the exponent γ^{-1} for $N=2$ at various fixed a is presented in Fig. 3. The parameter a ranges from 0.2 to 1.5 with step 0.1. Distinct oscillations correspond to the small values of a and b. As a grows, the behavior of the curves becomes smoother and the extended horizontal interval appears, reaching its maximum length at about $a=0.5$. Further increase of a causes the stability interval to get shorter and causes the rapid growth of the curves at its boundaries. It is essential that for all values of a the horizontal parts of the curves are at the same level. Averaging the results of the processing over the stability domain gives the

TABLE II. Critical exponents for the cubic model in three dimensions from the five-loop approximation in ϵ for some N.

N	η	ν	γ	γ_{sc}	$\frac{\gamma-\gamma_{sc}}{\gamma_{sc}}\times 100\%$
2	0.0350 ± 0.0003	0.6277 ± 0.0010	1.2358 ± 0.0040	1.2334	0.19%
3	0.0375 ± 0.0005	0.6997 ± 0.0024	1.3746 ± 0.0020	1.3732	0.10%
4	0.0365 ± 0.0005	0.7225 ± 0.0022	1.4208 ± 0.0030	1.4186	0.15%
5	0.0358 ± 0.0004	0.7290 ± 0.0016	1.4305 ± 0.0040	1.4319	0.10%
6	0.0354 ± 0.0003	0.7301 ± 0.0016	1.4322 ± 0.0040	1.4344	0.15%
∞	0.0350 ± 0.0003	0.7108 ± 0.0010	1.3993 ± 0.0020	1.3967	0.19%

number that we adopt as γ^{-1}. The accuracy for this approximate value is determined through the dispersion due to the variation of a and b. Similar behavior of the curves was observed for other critical exponents. Applying to them the developed algorithm, we obtain the critical exponent values for the cubic model for various N listed in Table II.

The case $N=\infty$ corresponds to the Ising model with equilibrium magnetic impurities. In this limit the Ising critical exponents α and ν are renormalized according to Fisher [13]. The values of η, ν, and γ were computed by processing original series for η, ν^{-1}, and γ^{-1}, while γ_{sc} was found by the scaling: $\gamma_{sc}=\nu(2-\eta)$. It is seen from the table that γ_{sc} differs from γ by no more than 0.2%, and this may confirm the good accuracy of the estimates obtained.

Let us compare our results with earlier estimates of the critical exponents for the pure Ising model. It is known that due to the symmetry of the initial Hamiltonian of the cubic model at $N=2$, the critical exponents for the cubic and the Ising fixed points coincide. The following estimates were found for the Ising model [8]: $\eta=0.035\pm0.002$, $\nu=0.628\pm0.001$. These numbers are in excellent agreement with the data of Table II at $N=2$. Our estimates are also in good agreement with the results of Ref. [10], where a substantially different method was employed.

In conclusion, let us formulate the results of the present paper. An approach to summation of divergent series has been suggested. The method employs the Borel transformation combined with a conformal mapping. It relies upon the stability of the result of processing on the transformation parameters and therefore does not require knowing the exact asymptotic behavior of the series. The method has been tested on the functions expanded in their asymptotic series and applied to estimating critical exponent values for the cubic model. The principal observation is that within our approach, summation of the perturbative series of both simple and complex (anisotropic) models exhibits the same behavior. This allows one to apply the developed technique to process divergent series arising in a number of anisotropic models describing phase transitions in real substances [14]. It can be expected that the proposed summation method may be useful in other fields of physics (e.g., QCD and QED) where one deals with divergent series, but conventional resummation techniques are inapplicable.

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